Homological Method in Quantum Field Theory

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Spring 2022

Introduction

Physics System:

$$S : \mathcal{E} \longmapsto \mathbb{R}$$

action functional   space of fields

Classical Physics: $\mathrm{Crit}(S) = \{\delta S = 0\}/\sim$

Quantum Physics: $\int_{\mathcal{E}} e^{\frac{i}{\hbar}S}$ = "path integral"

Eg: ① $\mathcal{E} = C^\infty(X)$ scalar field theory

$$S[\phi] = \int_X |d\phi|^2 + m^2\phi^2 \qquad \phi \in C^\infty(X)$$

② $\mathcal{E} = \left\{ \text{connections on } \begin{array}{c} V \\ \downarrow \\ X \end{array} \right\}$ gauge theory

$$YM[A] = \int \mathrm{Tr}\, F \wedge * F \qquad F = dA + \frac{1}{2}[A, A]$$

$$CS[A] = \frac{1}{2}\int \mathrm{Tr}\, A \wedge dA + \frac{1}{6}\int \mathrm{Tr}\, A \wedge [A, A]$$

$(\dim X = 3)$

③ $\mathcal{E} = \{ \text{map } \Sigma \mapsto X \}$ σ-model

④ $\mathcal{E} = \{ \text{metrics on } X \}$ gravity

- $\mathcal{E}$ is BIG, $\int_{\mathcal{E}}$ ∞-dim integral has no rigorous definitions in general
- $\hbar$-asymptotic theory exists: perturbative renormalization theory

Observables

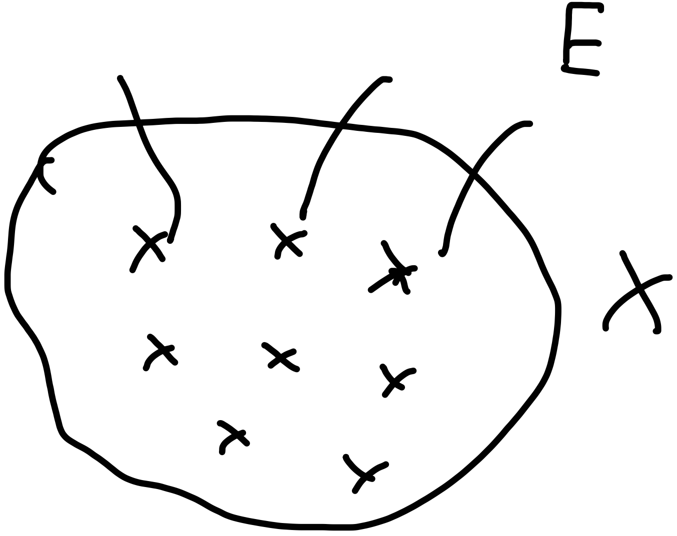
Suppose we consider a QFT on X

X: spacetime $\qquad \mathcal{E} = \Gamma(X, E)$ fields

we want to understand $\int_{\mathcal{E}}$

① $X = pt$. $\quad \mathcal{E} = \mathbb{R}^n \rightsquigarrow$ Calculus.

② $\dim X > 0$.



$$\mathcal{E} \neq \prod_{p \in X} E_p$$

$\rightsquigarrow$ topology of X makes a difference

new structures $\Longrightarrow$ Observable algebras

Roughly speaking

$$\text{observables} = \text{functions on fields}$$

$$= \mathcal{O}(\mathcal{E}) \text{ (or certain homologies)}$$

<u>Eg</u>: linear observable = distributions

New structures come from the following fact:

given $\mathcal{U} \subset X$ open, we can talk about

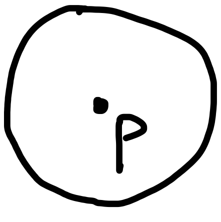
$$\mathrm{Obs}(\mathcal{U}) = \text{observables supported in } \mathcal{U}.$$

<u>Eg</u>: $\mathcal{E} = C^\infty(X)$, $p \in X$. Consider

$$\theta_1 : \mathcal{E} \longmapsto \mathbb{R}$$

$$\theta_1(f) = f(p)^m \quad \forall f \in \mathcal{E} = C^\infty(X)$$

θ_1 is an observable supported in any open nbhd of p.



Let $\mathcal{E}(U) = \Gamma(U, E)$ then

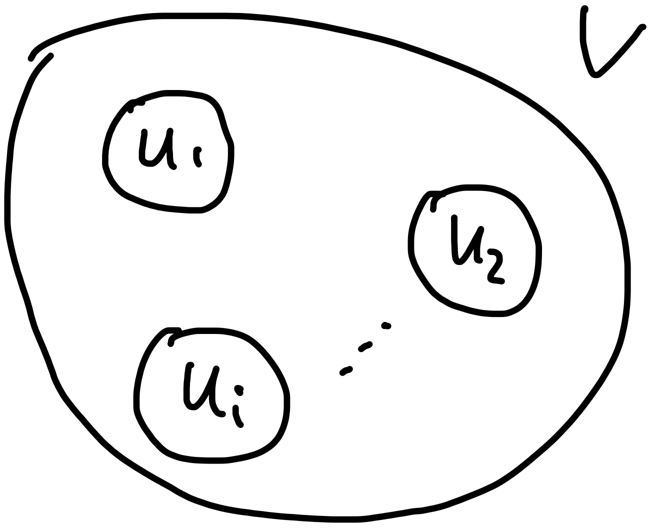
$$Obs(U) = \text{functions on } \mathcal{E}(U).$$

The new structure is the

factorization product / operator product expansion (OPE)

Given disjoint open subset $U_i \subset V$

$$\coprod U_i \subset V$$



we have a map

(factorization product)

$$\bigotimes_i Obs(U_i) \longmapsto Obs(V)$$

Intuitively, $\mathcal{E}(V) \xrightarrow{\text{restriction}} \mathcal{E}(U_i)$

$$\Rightarrow \mathcal{O}(\mathcal{E}(U_i)) \longmapsto \mathcal{O}(\mathcal{E}(V))$$

if we "multiply" those "functions", we get

$$\bigotimes_i \mathrm{Obs}(\mathcal{U}_i) \longmapsto \mathrm{Obs}(V)$$

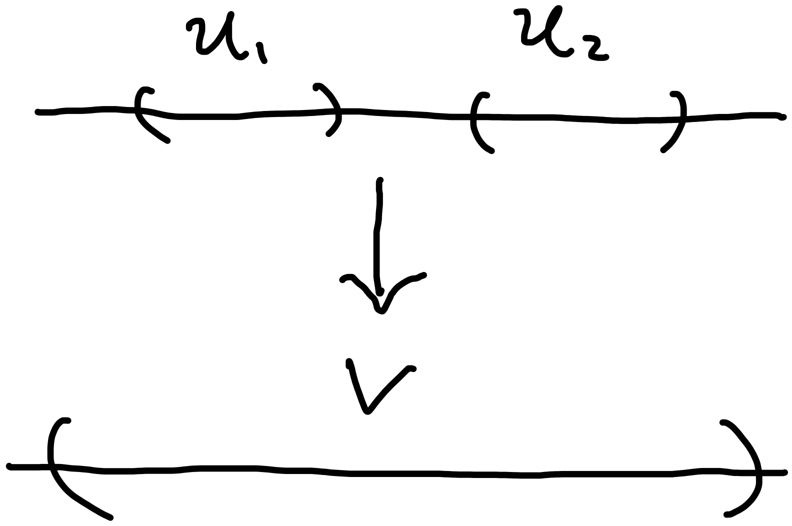
This requires further "quantum corrections"

fields in $\mathcal{U}_i$'s may "talk" to each other.

<u>Eg</u>. $\dim X = 1$ (topological quantum mechanics)

In top QFT, $\mathrm{Obs}(\mathcal{U})$ only depends on the topology of $\mathcal{U}$.

Consider $\dim X = 1$



$\mathrm{Obs}(\mathcal{U}) = A$ for $\mathcal{U}$ contractible

$$\Rightarrow \quad \begin{array}{ccccc} \mathrm{Obs}(\mathcal{U}_1) & \otimes & \mathrm{Obs}(\mathcal{U}_2) & \longmapsto & \mathrm{Obs}(V) \\ \parallel & & \parallel & & \parallel \\ A & \otimes & A & \longmapsto & A \end{array}$$

We find an associative algebra.

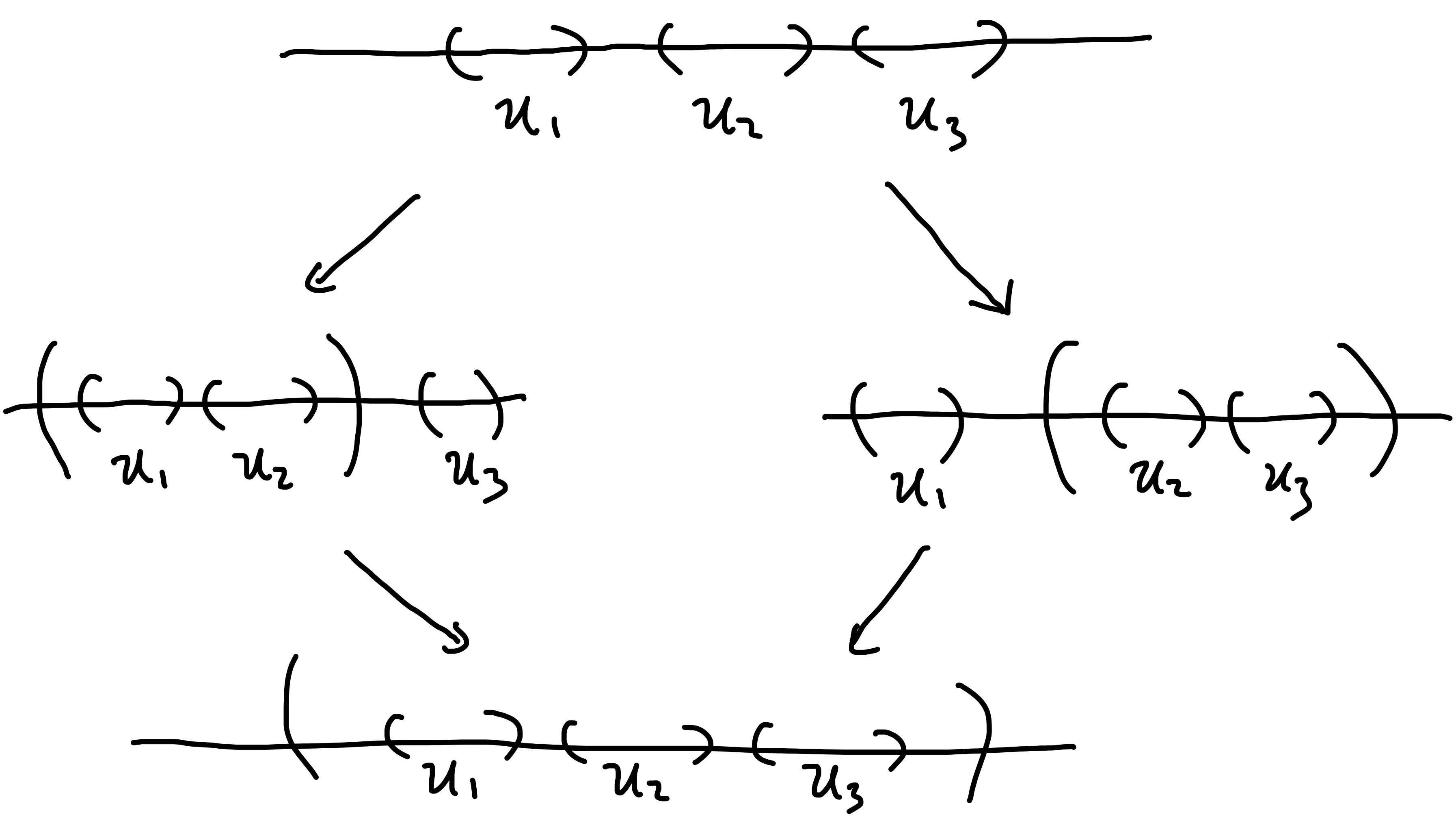
Algebraic Structure :

$$H_0(\mathbb{R}-\{0\}) = H_0(\mathbb{R}-\{0\})$$

$$= \mathbb{Z}\,\text{Left} \oplus \mathbb{Z}\,\text{Right}.$$

left / right multiplication

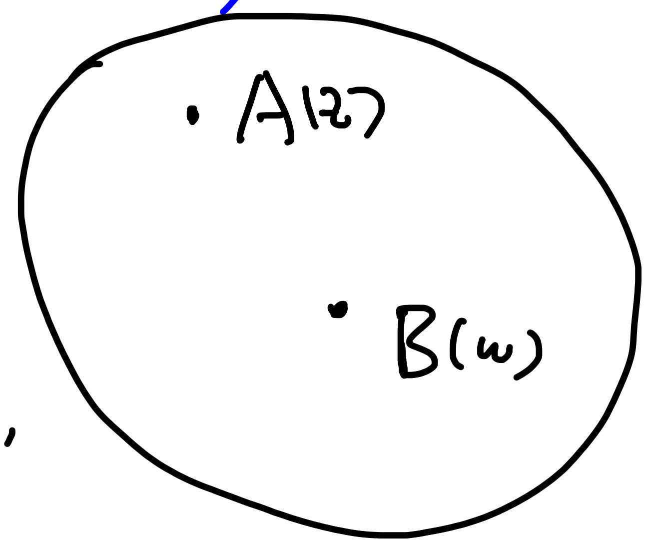
Associativity comes from further consistency



$$\Rightarrow \quad (a\cdot b)\cdot c = a\cdot(b\cdot c)$$

Eg. $\dim X = 2$ (Chiral QFT)

$$A(z)B(w) = \sum_{m\in\mathbb{Z}} \frac{(A_{(m)}B)(w)}{(z-w)^{m+1}}$$



We find ∞-many "product"

$$\{A_{(m)}B\}$$

observable algebras ⟹ vertex algebra

- Beilinson-Drinfeld
  - develop factorization algebra to formulate 2d Chiral CFT
  - Chiral homology
- Costello - Gwilliam
  construction of factorization algebras

from perturbative renormalization theory is the Batalin-Vilkovski (BV) formalism.

- BV formalism and Homological integration

$$\int = \text{Homology}$$

Calculus Revisited:

Let M be a compact oriented mfd of $\dim M = n$.

$(\Omega^\bullet(M), d)$ de Rham complex

$$\int_M : \Omega^\bullet(M) \longmapsto \mathbb{R}$$

$$\alpha \in \Omega^n(M) \longmapsto \int_M \alpha$$

Observe that $H^n_{dR}(M) = \mathbb{R}$

$$\Rightarrow \quad \int_M = H^n_{dR} \qquad \begin{aligned} \Omega^n(M) &\longrightarrow H^n_{dR}(M) \simeq \mathbb{R} \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

$$\Big\downarrow \, n \to \infty$$

$$\boxed{?}$$

BV approach

Define polyvector fields

$$PV^k(M) := \Gamma(M, \wedge^k TM)$$

$$PV^\cdot(M) = \bigoplus_k PV^k(M)$$

Let Ω be a fixed volume form on M. We can identify

$$PV^k(M) \longleftrightarrow \Omega^{n-k}(M)$$

$$\mu \longleftrightarrow \mu \lrcorner \Omega$$

Locally, if $\Omega = e^\varphi dx^1 \wedge \cdots \wedge dx^n$

$$\mu = \mu^{i_1 \cdots i_k} \partial_{i_1} \wedge \cdots \wedge \partial_{i_k}$$

then

$$\mu \lrcorner \Omega = \sum \pm \mu^{i_1 \cdots i_k} e^\varphi dx^1 \wedge \cdots \wedge \widehat{dx^{i_1}} \cdots \widehat{dx^{i_k}} \cdots dx^n$$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, phantom, sloped, "\cong"] & \Omega^1 \arrow[r, "d"] \arrow[d, phantom, sloped, "\cong"] & \cdots \arrow[r, "d"] & \Omega^n \arrow[dr, "\int"] & \\
PV^n \arrow[r, "\Delta"] & PV^{n-1} \arrow[r, "\Delta"] & \cdots \arrow[r, "\Delta"] & PV^0 \arrow[r, "\int_{BV}"'] & \mathbb{R}
\end{tikzcd}$$

$\Delta : PV^k \to PV^{k-1}$ divergence operator w.r.t. Ω

(Δ : BV operator)

Eg. Δ : $PV^1 = Vect(M) \longrightarrow PV^0 = C^\infty(M)$

the usual divergence.

$$\int_{BV} : PV^0 \longmapsto \mathbb{R} \qquad f \mapsto \int f\Omega$$

Homologically $\int_{BV} = H^0_{BV}$

- "dim M" does not appear
- In ∞-dim'l, renormalization helps to construct $\Delta \Longrightarrow$ Homological integration

Explicit form of Δ :

Locally in $\mathcal{U}$, let $\{x^i\}$ be local coordinates

$$\Omega = e^{f(x)} dx^1 \wedge \cdots \wedge dx^n$$

$$PV(u) = C^{\infty}(u)[\partial_1, \cdots, \partial_n] \quad \partial_i \partial_j = -\partial_j \partial_i$$

Let us write $\theta_i = \partial_i$, then $\mu \in PV(X)$ can be locally written as

$$\mu = \mu(x^i, \theta_i) \qquad \theta_i \theta_j = -\theta_j \theta_i$$

Let $\frac{\partial}{\partial \theta_i}$ be the derivative w.r.t. θ_i (from the left)

Eg: $\frac{\partial}{\partial \theta_1}(\theta_1 \theta_2) = \theta_2$

$$\frac{\partial}{\partial \theta_1}(\theta_2 \theta_1) = -\frac{\partial}{\partial \theta_1}(\theta_1 \theta_2) = -\theta_2$$

Then

$$\Delta = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \theta_i} + \sum_i (\partial_i f) \frac{\partial}{\partial \theta_i}$$

2nd order operator

Eg [Singularity Theory] Consider $\mathbb{C}^n$. Let

$$f: \mathbb{C}^n \longmapsto \mathbb{C}$$

be a polynomial w/. an isolated critical pt at 0

$$\mathrm{Crit}(f) = \{0\}$$

We consider holomorphic/polynomial polyvector fields

$$\mathcal{A} = \mathbb{C}[z^i, \theta_i] \qquad \theta_i \theta_j = -\theta_j \theta_i$$

Let $\Delta = \hbar \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \theta_i} + \sum_i (\partial_i f) \frac{\partial}{\partial \theta_i}$

- Observable $\mathrm{Obs}^q = H^\bullet(\mathcal{A}[[\hbar]], \Delta)$

  $\simeq$ Brieskorn lattice

- $\hbar$-filtration = Hodge filtration

- BV-integration $\rightsquigarrow$ $\langle \mathcal{O} \rangle = \int_{\mathcal{L}} \mathcal{O} e^{f/\hbar}$

  oscillatory integral $\quad$ Lefschetz thimble

Ref today: Effective Batalin-Vilkovisky quantization and geometric applications. arXiv:1709.00669